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A CALCULATION PROCEDURE FOR STEADY TWO-DIMENSIONAL ELLIPTIC FLOWS

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SUMMARY

A calculation procedure is presented for predicting steady two-dimensional elliptic flows. The method introduces a density correction concept and an algebraic equation for the velocity correction instead of the troublesome pressure correction equation in the SIMPLER procedure. Computations show that the method has the same rate of convergence as SIMPLER while saving about 20% computational effort per iteration. Although the method is described for steady two-dimensional situations, its extension to three-dimensional problems is very straightforward.

KEY WORDS Calculation procedure Density correction Elliptic flow

1. INTRODUCTION

The basic equations for fluid flow are the continuity equation and the momentum equations, which can be written for steady two-dimensional flow as

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0, \tag{1}$$

$$\frac{\partial}{\partial x}(\rho uu)+\frac{\partial}{\partial y}(\rho vu)=\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)-\frac{\partial p}{\partial x}+S^{u},$$

$$\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho vv)=\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)-\frac{\partial p}{\partial y}+S^{v}. \tag{2}$$

The above equations are non-linear and interlinked. The computation of fluid flow has to be an iteration procedure.

The main difficulty in fluid flow computation lies in the prediction of the pressure field. The pressure gradient appears in the momentum equation as an important source term, which is not expressible in terms of the velocity components or other variables. What indirectly determines the pressure field is the continuity equation. This indirect specification is not convenient as a computational procedure; a direct method of determining the pressure field and satisfying the continuity equation must be found.

Because of the difficulty in determining the pressure field, some vorticity-based methods were introduced in the 1960s.[1] The main features of these methods are deriving a vorticity transport equation by eliminating the pressure term from the momentum equation, together with the streamfunction, making the well known ω–ψ method.

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The ω–ψ method has several attractive features: the pressure term no longer appears and the coupled equations are reduced to two (ω and ψ). This method also brings some problems, such as the determination of the vorticity on solid walls, etc. The main problem is that the ω–ψ method cannot be extended to three-dimensional situations conveniently. Because most real problems are three-dimensional, the ω–ψ method has thus been strongly limited. As an alternative, primitive variable methods are widely used.

In 1972, Patankar and Spalding described a primitive variable calculation procedure for three-dimensional parabolic flows, which has been given the name SIMPLE (semi-implicit method for pressure-linked equations).[2] Its main features are deriving a pressure correction equation from the discretization continuity equation and momentum equations over a staggered grid system by omitting the term $\sum a_{nb}u'_{nb}$, then correcting the velocity and pressure fields to make them satisfy mass and momentum conservation simultaneously.

The omission in deriving the pressure correction equation does not influence the correctness of the final solution in the iteration procedure. The rate of convergence, however, is influenced by the omission. The approximate p' equation tends to overestimate the value of p', so that the SIMPLE procedure is prone to divergence unless some under-relaxation is used. Although the SIMPLE procedure has been successfully used for a number of problems, its rate of convergence has not always been satisfactory. A revised version, SIMPLER, has been formulated.[3,4] It takes advantage of the property that although the values of p' may be overestimated, the associated velocity corrections are of the right magnitude. In SIMPLER, therefore, the p' equation is used for the purpose of correcting only the velocity field, while a separate equation, the pressure equation, is used for evaluating the pressure.

No approximation such as omitting the term $\sum a_{nb}u'_{nb}$ was used in deriving the pressure equation. So the pressure field obtained from the pressure equation is likely to be more correct than the field constructed from the p' values. It is from this characteristic of the pressure equation that the SIMPLER procedure gives faster convergence and better iteration stability.

Although one iteration of SIMPLER requires about 30% more computational effort than one iteration of SIMPLE, the extra effort per iteration is amply compensated by a reduction in the number of iterations required for convergence. The SIMPLER procedure has been successfully used for many problems. However, four discretization equations have to be solved to obtain three primitive variables in SIMPLER. There must be something unnecessary. A better method is needed.

2. IMPLE PROCEDURE

A new calculation procedure named IMPLE (implicit method for pressure-linked equations) has been developed. It takes advantages of the pressure equation in the SIMPLER procedure: the pressure equation is used to obtain the pressure field while an algebraic equation is used to correct the velocity field.

2.1. Generation of the method

The discretized continuity equation and momentum equations for the control volumes shown in Figure 1 can be written as

$$(\rho uA)_w-(\rho uA)_e+(\rho vA)_s-(\rho vA)_n=0, \tag{3}$$

$$a_e u_e=\sum a_{nb}u_{nb}+b^u+A_e(p_p-p_e),$$

$$a_n v_n=\sum a_{nb}v_{nb}+b^v+A_n(p_p-p_n), \tag{4}$$

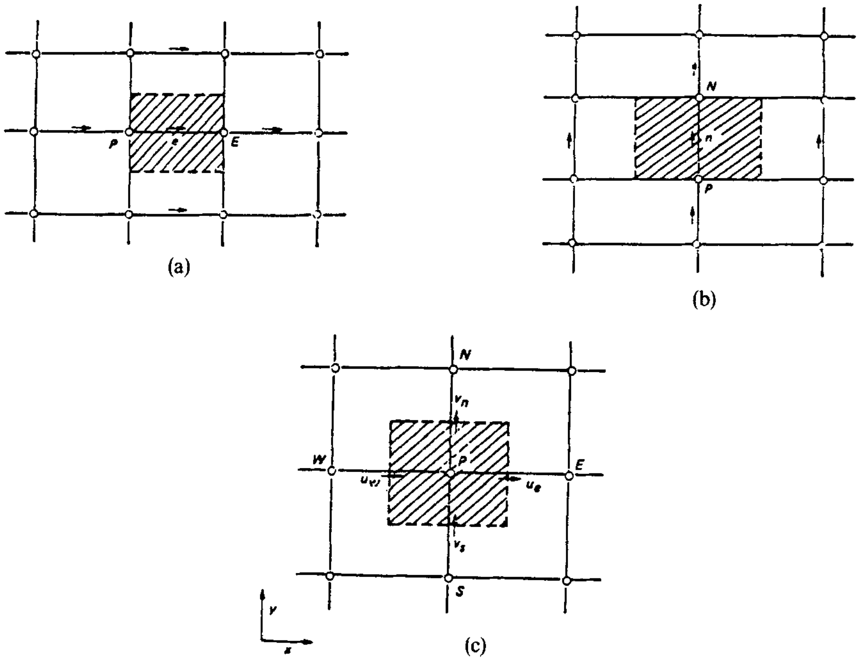

Figure 1. (a) Control volume for u (b) Control volume for v (c) Control volume for b

where the a coefficients contain mass fluxes, viscosities, etc., the term b includes the source terms other than the pressure gradient, and the As are the areas of the control volume.

Equations (4) can be written as

$$u_e = \hat{u}_e + d_e(p_p - p_e),$$
$$v_n = \hat{v}_n + d_n(p_p - p_n), \tag{5}$$

where the pseudovelocities $\hat{u}_e$ and $\hat{v}_n$ are given by

$$\hat{u}_e = \frac{\sum a_{nb} u_{nb} + b^u}{a_e},$$
$$\hat{v}_n = \frac{\sum a_{nb} v_{nb} + b^v}{a_n}, \tag{6}$$

and

$$d_e = A_e / a_e, \qquad d_n = A_n / a_n. \tag{7}$$

With equations (4) and (5), a pressure equation can be derived:

$$a_p p_p = a_e p_e + a_w p_w + a_n p_n + a_s p_s + b^p, \tag{8}$$

where

$$a_e = (\rho d A)_e, \qquad a_w = (\rho d A)_w, \qquad a_n = (\rho d A)_n,$$
$$a_s = (\rho d A)_s, \qquad a_p = a_e + a_w + a_n + a_s, \tag{9}$$
$$b^p = (\rho \hat{u} A)_w - (\rho \hat{u} A)_e + (\rho \hat{v} A)_s - (\rho \hat{v} A)_n. \tag{10}$$

Equation (8) is the pressure equation in the SIMPLER procedure, from which a pressure field p^* can be obtained for a 'guessed' velocity field. Let u^* and v^* denote the velocity field for p^*; we have

$$a_e u_e^* = \sum a_{nb} u_{nb}^* + b^u + A_e(p_P^* - p_e^*),$$
$$a_n v_n^* = \sum a_{nb} v_{nb}^* + b^v + A_n(p_P^* - p_n^*). \quad (11)$$

In general, u^* and v^* will not satisfy the continuity equation. This implies

$$b = (\rho u^* A)_w - (\rho u^* A)_e + (\rho v^* A)_s - (\rho v^* A)_n \neq 0. \quad (12)$$

The 'starred' u^* and v^* must be corrected.

Suppose that the correct velocity field can be described as

$$u_e = u_e^* + u_e', \qquad v_n = v_n^* + v_n', \quad (13)$$

where u_e' and v_n' are named velocity corrections.

Construct a 'time-dependent' term A; let

$$A - b = 0 \quad (14)$$

and

$$A = \rho_P' \mathrm{d}V/\mathrm{d}t \quad (15)$$

where ρ_P' is defined as the density correction, $\mathrm{d}V$ is the control volume and $\mathrm{d}t$ is the time step in the iteration procedure. Substituting equation (15) into equation (14), we have

$$\rho_P' = b\,\mathrm{d}t/\mathrm{d}V. \quad (16)$$

A density correction field can be obtained from equations (12) and (16). It may be noted that the density correction is likely to be a 'potential' of mass transfer: $\rho' > 0$ means a positive velocity correction; $\rho' < 0$ means a negative velocity correction; and $\rho' = 0$ implies satisfaction of the continuity equation. The density correction ought to make its contribution to the velocity correction. So we let

$$u_e' = (\rho_P' - \rho_e')\,|u_e^*|/\rho_e^*, \qquad v_n' = (\rho_P' - \rho_n')\,|v_n^*|/\rho_n^*, \quad (17)$$

where ρ^* is an approximate density corresponding to the u^*, v^* solution. For incompressible flows, ρ^* is a constant.

Thus a velocity correction formula is generated by substituting equation (17) into equation (13):

$$u_e = u_e^* + (\rho_P' - \rho_e')\,|u_e^*|/\rho_e^*, \qquad v_n = v_n^* + (\rho_P' - \rho_n')\,|v_n^*|/\rho_n^*. \quad (18)$$

Now the IMPLE procedure can be described as follows:

1. Guess a velocity field.
2. Calculate the coefficients in the momentum equations (4) and hence obtain the pseudo-velocities $\hat{u}$ and $\hat{v}$ from equations (6).
3. Solve the pressure equation (8) to get p.
4. Regarding this pressure field as p^*, solve the momentum equations (11) to obtain u^* and v^*.
5. Calculate the density correction ρ' by equation (12) and correct the starred velocities with equations (18).
6. Return to step 2 with the corrected velocity field and repeat the procedure until convergence.

2.2. *Discussion of IMPLE*

The procedure has been given the name IMPLE because it does not use the semi-implicit p' equation. For the same reason, the IMPLE procedure requires much less computational effort and computer storage than SIMPLER does. The iteration stability of IMPLE is satisfactory since the better pressure equation is used to obtain the pressure field.

It may be noted that the expression for b implies that b is the 'mass source' present in the starred velocity field. The task of the density correction is to annihilate this mass source.

The IMPLE procedure is an iterative one. The continuity equation and momentum equations are satisfied step by step in the iteration procedure. The time step $\mathrm{d}t$ in equation (16) determines the magnitude of the velocity correction. The value of $\mathrm{d}t$ should be chosen to make the values of u', v' of the same order as or less than those of u^*, v^*. Larger $\mathrm{d}t$ gives faster convergence and smaller $\mathrm{d}t$ results in slower convergence.

3. TEST CALCULATIONS

3.1. Statement of the problem

The IMPLE procedure was tested and compared with SIMPLE and SIMPLER by application to a steady two-dimensional flow in a square cavity with a moving wall (Figure 2). The flow is regarded as laminar; the fluid properties and temperature are taken to be uniform.

3.2. Computation details

The powder-law scheme was used for the flux expression and the line-by-line technique for the solution of the discretization equations.

The computations were performed on an IBM-4341 computer. A uniform rectangular grid of 21×21 nodes and a time step $\mathrm{d}t = 0{\cdot}05$ were used for all runs.

3.3. Computation results

Comparison of CPU times. The CPU times per iteration in the IMPLE, SIMPLE and SIMPLER procedure are respectively 0·993, 0·936 and 1·20 s. This means that the IMPLE procedure needs 6·1% more computational effort per iteration than SIMPLE and 21% less than SIMPLER.

Iteration history. Figure 3 shows the iteration histories of IMPLE, SIMPLE and SIMPLER for $Re = 10$, 100, 500 and 1000. The rate of convergence of IMPLE is much faster than SIMPLE and of the same order as SIMPLER.

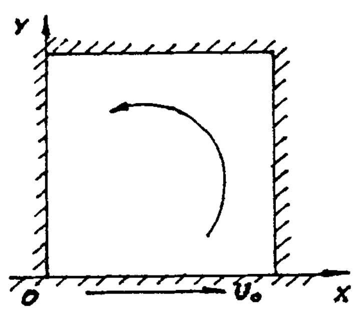

Figure 2. Cavity flow

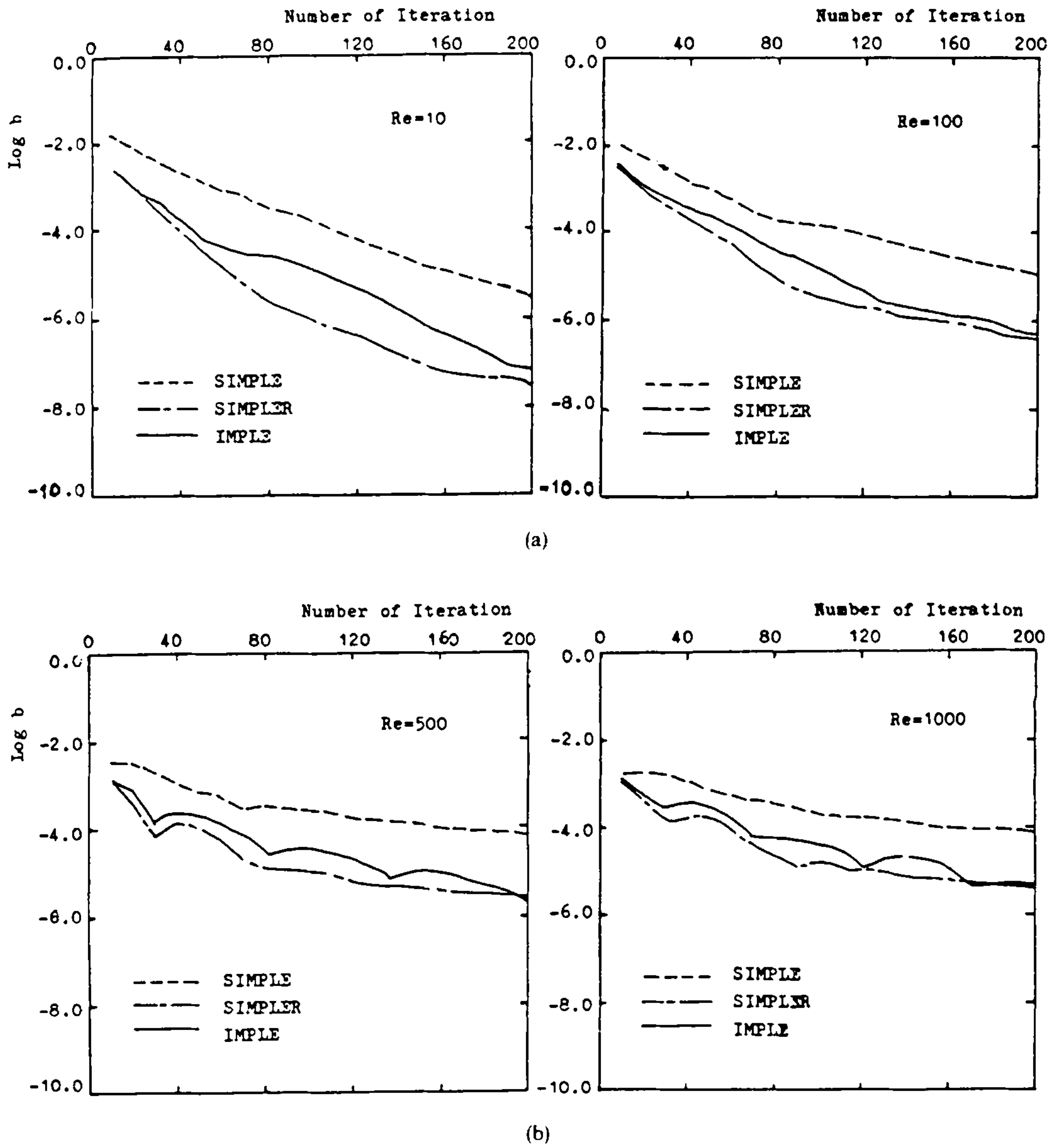

Figure 3. Iteration history: comparisons of IMPLE, SIMPLE and SIMPLER

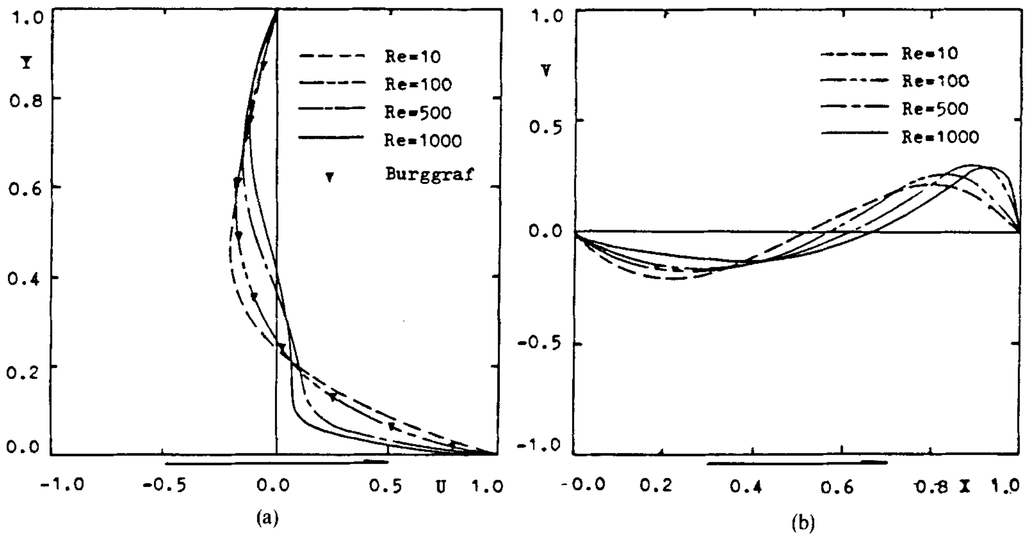

Figure 4. Centreline velocities: (a) u on vertical centreline; (b) v on horizontal centreline

Velocity field. Figure 4 shows the variation of centreline velocities for various velocities of the moving wall. The results of IMPLE, SIMPLE and SIMPLER are in complete coincidence. The predictions for $Re = 100$ are compared with the numerical results of Burggraf;[5] the agreement is very good.

Streamline patterns. Figure 5 shows the streamline patterns computed from the velocity field by the definition of the streamfunction. The results of IMPLE and SIMPLER are exactly the same, while the results of SIMPLE are a little different because the degree of convergence is different.

4. CONCLUSIONS

The present paper has described a generally applicable, accurate and economical method for calculating steady two-dimensional flows. The method introduces a density correction concept and an algebraic equation for the velocity correction instead of the troublesome pressure correction equation, and hence requires much less computational effort and computer storage than the SIMPLER procedure does.

Computations show that this method requires the same computational effort and computer storage as SIMPLE and has the same rate of convergence as SIMPLER does. Although the method is described for steady two-dimensional flows, its extension to three-dimensional situations is very straightforward.

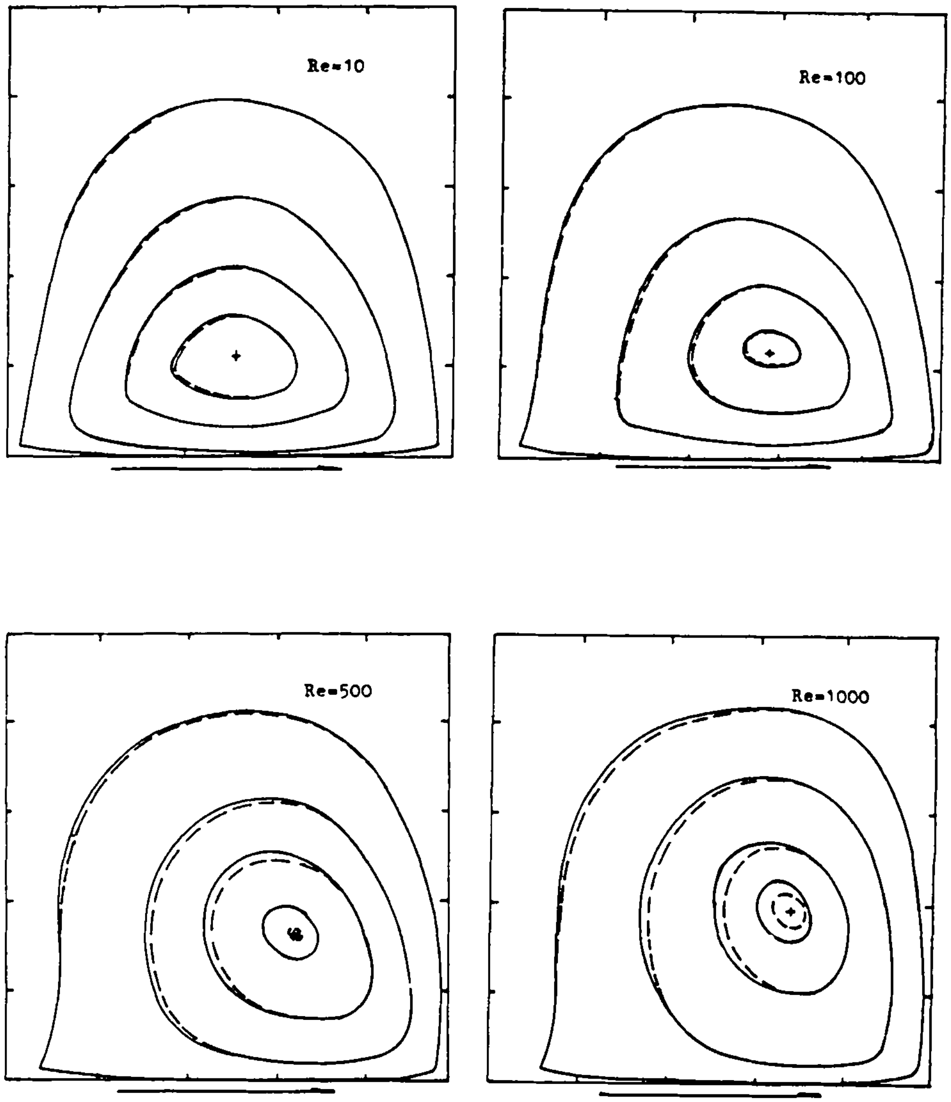

Figure 5. Streamline patterns:——, IMPLE and SIMPLER; -----, SIMPLE

APPENDIX: NOMENCLATURE

a	discretization coefficient
A	area of a control volume face; also time-dependent term
b	constant term in discretization equation
d	pressure coefficient
$\mathrm{d}t$	time step
$\mathrm{d}V$	control volume
p	pressure
p^*	estimated pressure
p'	pressure correction

Re	Reynolds number
S	source term
u, v	velocity components in x- and y-direction
u^*, v^*	velocities based on p^*
u', v'	velocity corrections
$\hat{u}$, $\hat{v}$	pseudovelocities
x, y	Cartesian co-ordinates
μ	viscosity
ρ, ρ^*	densities
ρ'	density correction
ω	vorticity
ψ	streamfunction

Subscripts

e, n, s, w	control volume faces; also grid points
nb	neighbouring grid points
p	central grid point

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